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GROUND STATES OF ONE-DIMENSIONAL LONG-RANGE FERROMAGNETIC ISING MODEL WITH EXTERNAL FIELD

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A zero-temperature phase-diagram of the one-dimensional ferromagnetic Ising model is investigated. It is shown that at zero temperature spins of any compact collection of lattice points with identically oriented external field are identically oriented.

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1. Introduction

Consider the one-dimensional ferromagnetic Ising model with long range interaction:

$$H(\phi) = -\sum_{x,y\in\mathbf{Z}^1;x>y} U(x-y)\phi(x)\phi(y) - \sum_{x\in\mathbf{Z}^1} h_x\phi(x)\,, \tag{1}$$

where spin variables $\phi(x)$ associated with the one-dimensional lattice sites x take values -1 and $+1$, the pair potential $U(x-y) = (x-y)^{-\gamma}$, $\gamma > 1$ and $\{h_x, x \in \mathbf{Z}^1\}$ is an external field. The condition $\gamma > 1$ is necessary for the existence of the thermodynamical limit. If $\gamma > 2$ then $\sum_{x\in\mathbf{Z}^1,x>0} xU(x) < \infty$ and the model (1) has a unique Gibbs state[1–3] at any non-zero value of the temperature.

In the absence of the external field ($h_x \equiv 0$) the model exhibits a phase transition. Suppose that a positive decreasing potential $U(r) = U(|x-y|)$ satisfies the conditions: $\sum_{r=1}^{\infty} U(r) < \infty$ and $\sum_{r=1}^{\infty}(\ln\ln(r+4))\,(r^3U(r))^{-1} < \infty$ (the model (1) with $1 < \gamma < 2$ readily satisfies these conditions). Then there exists a value of the inverse temperature β_1 such that if $\beta > \beta_1$ then there are at least two extreme Gibbs states P^+ and P^- corresponding to the ground states $\phi(x) = +1$ and $\phi(x) = -1$.[4,5] This result is related to the phenomenon of surface tension in one-dimensional models. The phase transition also takes place in the borderline case

$\gamma = 2$.[6–8] In the case of unbiased random external field the model (1) exhibits low temperature phase transition with probability one.[9]

In this paper we investigate the phase diagram of the model (1) under nonzero external fields.

The following two results show that if the absolute value of the external field is sufficiently strong or the aligned vectors h_x are "well organized" and constitute long blocks, then the external field exterminates the pair interaction and the dependence on the boundary conditions disappears in the limit:

Theorem 1.[10] *At any fixed value of the inverse temperature β there exists a constant h_0 such that for all realizations of the external field $\{h_x, x \in \mathbf{Z}^1\}$ satisfying $|h_x| > h_0$, $x \in \mathbf{Z}^1$ the model* (2) *has at most one limiting Gibbs state.*

Theorem 2.[11] *Let h_x be a periodic function of period $2r$: $h_x^r = h_{x+2rk}$ for all integer values of k and for some sufficiently small fixed positive ϵ.*

$$h_x^r = \begin{cases} +\epsilon & \text{if } x = 1, \ldots, r\,, \\ -\epsilon & \text{if } x = r+1, \ldots, 2r\,. \end{cases}$$

There exist natural numbers $R_1 = R_1(\epsilon)$ and $R_2 = R_2(\epsilon)$ such that at all sufficiently small temperatures the model (1) has at least two limiting Gibbs states for all $r \le R_1$ and at most one limiting Gibbs state for all $r > R_2$.

Thus, if the absolute value of the external field is not sufficiently strong and the aligned vectors h_x do not constitute long blocks then the set of limiting Gibbs states of the model (1) may have one or more than one element.[11]

2. The Structure of Ground States

In this section, we investigate the set of ground states of the model (1), where $h > 0$ and $h_x = \pm h$.

A configuration ϕ^{gr} is said to be a ground state of the model (2), if for any finite set $A \subset \mathbf{Z}^1$ $H(\phi') - H(\phi^{gr}) \ge 0$, where ϕ' is a perturbation of ϕ^{gr} on the set A.

Let us define a configuration φ_h by $\varphi_h(x) = \mathrm{sign}(h_x)$. The ground state configuration ϕ^{gr} results ferromagnetic "struggle" between spins of φ_h.

For integer a and b, $[a, b]$ denotes the set of all integers lying between a and b, including a and b. Let ϕ be any fixed configuration. We say that an interval $[a, b]$ with integer endpoints is a block if $\varphi([a, b])$ is a constant and for any other interval $[c, d]$ with integer endpoints satisfying $[a, b] \subset [c, d]$, $\varphi([c, d])$ is not a constant (the case $a = b$ is not excluded). The collection of all blocks of ϕ we order in the following way: $0 \in [a_0, b_0]$ and for each integer i $b_i + 1 = a_{i+1}$. Thus, $Z^1 = \bigcup_{\infty < i < \infty} [a_i, b_i]$. The block $[a_i, b_i]$ of the configuration ϕ we denote by $\Delta_i(\phi)$.

The blocks $\Delta(\phi_h)$ of the configuration φ_h we call h-blocks.

It is known that that the restriction of a ground state to any "long" block coincides with φ_h: There is a constant L such that for all h-blocks $\Delta(\phi_h)$ with lengths exceeding L, $\phi^{gr}(\Delta(\phi_h)) = \varphi_h(\Delta(\phi_h))$.[11]

It turns out that the restriction of a ground state to any block is a constant configuration.

Theorem 3. *Let ϕ^{gr} be a ground state of the model* (1) *with arbitrary decreasing pair potential $U(\cdot)$ and $\Delta(\phi_h)$ be a h-block. Then $\phi^{gr}(\Delta(\phi_h)) = \varphi_h(\Delta(\phi_h))$ or $\phi^{gr}(\Delta(\phi_h)) = -\varphi_h(\Delta(\phi_h))$.*

Proof. Suppose that there is a h-block $\Delta(\phi_h) = [a_k, b_k]$ such that $\phi^{gr}(\Delta(\phi_h)) \neq \varphi_h(\Delta(\phi_h))$ and $\phi^{gr}(\Delta(\phi_h)) \neq -\varphi_h(\Delta(\phi_h))$. Then without loss of generality, we can suppose that $\varphi_h([a_k, b_k]) \equiv 1$ and there are two points z_0 and z_0+1 both from $[a_k, b_k]$ such that $\phi^{gr}(z_0) = 1$ and $\phi^{gr}(z_0+1) = -1$. Let $\{\Delta_l(\phi^{gr}); l \in \mathbf{Z}^1\}$ be the set of all blocks of ϕ^{gr}. By definitions, z_0 and z_0+1 belong to two neighboring blocks of ϕ^{gr}: for some l_0 there are blocks $\Delta_{l_0}(\phi^{gr}) = [a_{l_0}, b_{l_0}]$ and $\Delta_{l_0+1}(\phi^{gr}) = [a_{l_0+1}, b_{l_0+1}]$ such that $b_{l_0} = z_0$ and $a_{l_0+1} = z_0 + 1$. □

Let us define two configurations:

$$\widetilde{\phi}^{gr}_{z_0} = \begin{cases} -\phi^{gr} & \text{if } x = z_0\,, \\ \phi^{gr} & \text{if } x \neq z_0\,, \end{cases}$$

and

$$\widetilde{\phi}^{gr}_{z_0+1} = \begin{cases} -\phi^{gr} & \text{if } x = z_0+1\,, \\ \phi^{gr} & \text{if } x \neq z_0+1\,. \end{cases}$$

Since ϕ^{gr} is a ground state, $H(\widetilde{\phi}^{gr}_{z_0}) - H(\phi^{gr}) \geq 0$ and $H(\widetilde{\phi}^{gr}_{z_0+1}) - H(\phi^{gr}) \geq 0$. Therefore,

$$H(\widetilde{\phi}^{gr}_{z_0}) - H(\phi^{gr}) + H(\widetilde{\phi}^{gr}_{z_0+1}) - H(\phi^{gr}) \geq 0\,. \tag{2}$$

It can be readily seen that

$$H(\widetilde{\phi}^{gr}_{z_0}) - H(\phi^{gr}) = 2h + 2\sum_{i=0}^{\infty}(-1)^i \sum_{x\in\Delta_{l_0-i}} U(b_{l_0} - x) + 2\sum_{j=1}^{\infty}(-1)^j \sum_{x\in\Delta_{l_0+j}} U(x - b_{l_0})\,, \tag{3}$$

and

$$H(\widetilde{\phi}^{gr}_{z_0+1}) - H(\phi^{gr}) = -2h + 2\sum_{i=0}^{\infty}(-1)^{i+1} \sum_{x\in\Delta_{l_0-i}} U(a_{l_0+1} - x) + 2\sum_{j=1}^{\infty}(-1)^{j+1} \sum_{x\in\Delta_{l_0+j}} U(x - a_{l_0+1})\,. \tag{4}$$

The sum of (3) and (4) after canceling of opposite terms yields

$$H(\widetilde{\phi}^{gr}_{z_0}) - H(\phi^{gr}) + H(\widetilde{\phi}^{gr}_{z_0+1}) - H(\phi^{gr}) = 2\sum_{i=0}^{\infty}(-1)^{i+1} U(a_{l_0+1} - a_{l_0-i}) + 2\sum_{j=1}^{\infty}(-1)^{j} U(b_{l_0+j} - b_{l_0})\,.$$

Now by pairing of neighboring terms in both sums we get

$$H(\widetilde{\phi}^{gr}_{z_0}) - H(\phi^{gr}) + H(\widetilde{\phi}^{gr}_{z_0+1}) - H(\phi^{gr}) = 2\sum_{i=0}^{\infty} A_i + 2\sum_{j=1}^{\infty} B_i\,,$$

where $A_i = U(a_{l_0+1} - a_{l_0-2i-1}) - U(a_{l_0+1} - a_{l_0-2i})$ and $B_j = U(b_{l_0+2j} - b_{l_0}) - U(b_{l_0+2j-1} - b_{l_0})$.

Finally we note that since the potential function $U(\cdot)$ is decreasing, all terms A_i and B_j of both sums are negative and consequently $H(\widetilde{\phi}^{gr}_{z+1}) - H(\phi^{gr}) + H(\widetilde{\phi}^{gr}_{z}) - H(\phi^{gr}) < 0$, which contradicts (2). The proof of Theorem 3 is completed.

By Theorem 3, any ground state ϕ^{gr} can be obtained by reversing of all spins of some not long h-blocks of ϕ_h. In other words, spins of ϕ^{gr} belonging to the same h-block are strongly correlated and change their values synchronously. This fact considerable simplifies the structure of the zero-temperature phase diagram of (1). The following theorem shows that if in a ground state spins of some h-block of ϕ_h are reversed, then the spins of its neighbor blocks are not reversed:

Theorem 4. *Let $\Delta_1(\phi_h), \Delta_2(\phi_h)$ and $\Delta_3(\phi_h)$ be three consecutive h-blocks. Suppose that $\phi^{gr}(\Delta_2(\phi_h)) = -\varphi_h(\Delta_2(\phi_h))$. Then $\phi^{gr}(\Delta_1(\phi_h)) = \varphi_h(\Delta_1(\phi_h))$ and $\phi^{gr}(\Delta_3(\phi_h)) = \varphi_h(\Delta_3(\phi_h))$.*

Proof. Without loss of generality we suppose that $h_x = -h$ for all $x \in \Delta_1(\phi_h)$, $x \in \Delta_3(\phi_h)$ and $h_x = h$ for all $x \in \Delta_2(\phi_h)$. Let $\Delta_1(\phi_h) = [a_{k-1}, b_{k-1}]$, $\Delta_2(\phi_h) = [a_k, b_k]$, $\Delta_3(\phi_h) = [a_{k+1}, b_{k+1}]$ and at least one of the relations $\phi^{gr}(\Delta_1(\phi_h)) = \varphi_h(\Delta_1(\phi_h))$; $\phi^{gr}(\Delta_3(\phi_h)) = \varphi_h(\Delta_3(\phi_h))$ is not held. Then without loss of generality we suppose that $\phi^{gr}(a_{k+1}) = 1$. Let $z_0 = b_k$ and $z_0 + 1 = a_{k+1}$. Thus, $\phi^{gr}(z_0) = -1$ and $\phi^{gr}(z_0 + 1) = 1$. Let $\{\Delta_l(\phi^{gr}); l \in \mathbf{Z}^1\}$ be the set of all blocks of ϕ^{gr}. By definitions, z_0 and $z_0 + 1$ belong to two neighboring blocks of ϕ^{gr}: for some l_0 there are blocks $\Delta_{l_0}(\phi^{gr}) = [a_{l_0}, b_{l_0}]$ and $\Delta_{l_0+1}(\phi^{gr}) = [a_{l_0+1}, b_{l_0+1}]$ such that $b_{l_0} = z_0$ and $a_{l_0+1} = z_0 + 1$. $\square$

As in the proof of Theorem 3, we define configurations $\widetilde{\phi}^{gr}_{z_0}$ and $\widetilde{\phi}^{gr}_{z_0+1}$. Since ϕ^{gr} is a ground state, (2) holds. It can be readily seen that

$$\begin{aligned} H(\widetilde{\phi}^{gr}_{z_0}) - H(\phi^{gr}) = -2h + 2\sum_{i=0}^{\infty}(-1)^i \sum_{x\in\Delta_{l_0-i}} U(b_{l_0} - x) \\ + 2\sum_{j=1}^{\infty}(-1)^j \sum_{x\in\Delta_{l_0+j}} U(x - b_{l_0})\,, \end{aligned} \tag{5}$$

and

$$H(\widetilde{\phi}^{gr}_{z_0+1}) - H(\phi^{gr}) = -2h + 2\sum_{i=0}^{\infty}(-1)^{i+1}\sum_{x\in\Delta_{l_0-i}} U(a_{l_0+1}-x)$$
$$+ 2\sum_{j=1}^{\infty}(-1)^{j+1}\sum_{x\in\Delta_{l_0+j}} U(x-a_{l_0+1})\,. \tag{6}$$

Again as in the proof of Theorem 3, we get that

$$H(\widetilde{\phi}^{gr}_{z_0}) - H(\phi^{gr}) + H(\widetilde{\phi}^{gr}_{z_0+1}) - H(\phi^{gr}) = -4h + 2\sum_{i=0}^{\infty}A_i + 2\sum_{j=1}^{\infty}B_i$$

where $A_i = U(a_{l_0+1} - a_{l_0-2i-1}) - U(a_{l_0+1} - a_{l_0-2i})$ and $B_j = U(b_{l_0+2j} - b_{l_0}) - U(b_{l_0+2j-1} - b_{l_0})$.

Now since the potential function $U(\cdot)$ is decreasing, all terms A_i and B_j of both sums are negative and consequently $H(\widetilde{\phi}^{gr}_{z+1}) - H(\phi^{gr}) + H(\widetilde{\phi}^{gr}_{z}) - H(\phi^{gr}) < -4h < 0$, which contradicts (2). The proof of Theorem 4 is completed.

In the case when the external field takes three values $\pm h$ and 0 one can define ϕ_h by

$$\phi_h(x) = \begin{cases} \mathrm{sign}(h_x) & \text{if } h_x \neq 0\,, \\ 0 & \text{if } h_x = 0\,, \end{cases}$$

and set h-blocks $\Delta(\phi)$ as above. In this case the structure of a ground state is also analogous to the above-mentioned ones:

Theorem 5. *Let ϕ^{gr} be a ground state of the model* (1) *and $\Delta(\phi_h)$ be a h-block. Then $\phi^{gr}(\Delta(\phi_h))$ is a constant.*

The proof is analogous to the proof of Theorem 3 and is omitted.

If the ground state is unique and stable then at sufficiently low values of the temperature the limiting Gibbs state is also unique.[12] In general, the investigation of the non-zero temperature phase diagram of the model (1) is a rather complicated problem.

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